

# Weak convergence of complex-valued measure for bi-product path space induced by quantum walk

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**Abstract.** In this paper, a complex-valued measure of bi-product path space induced by quantum walk is presented. In particular, we consider three types of conditional return paths in a power set of the bi-product path space (1)  $\Lambda \times \Lambda$ , (2)  $\Lambda \times \Lambda'$  and (3)  $\Lambda' \times \Lambda'$ , where  $\Lambda$  is the set of all  $2n$ -length ( $n \in \mathbb{N}$ ) return paths and  $\Lambda'(\subseteq \Lambda)$  is the set of all  $2n$ -length return paths going through  $nx$  ( $x \in [-1, 1]$ ) at time  $n$ . We obtain asymptotic behaviors of the complex-valued measures for the situations (1)-(3) which imply two kinds of weak convergence theorems (Theorems 1 and 2). One of them suggests a weak limit of weak values.

## 1 Introduction

Let the set of all the  $n$ -truncated paths be  $\Omega_n = \{-1, 1\}^n$ . Denote the coin space  $\mathcal{H}_C$  spanned by choice of direction at each time step, that is,  $\mathbf{e}_{-1} = {}^T[1, 0]$  and  $\mathbf{e}_1 = {}^T[0, 1]$ . Let quantum coin on  $\mathcal{H}_C$  be

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{U}(2)$$

with  $abcd \neq 0$ , where  $\mathrm{U}(2)$  is the set of two-dimensional unitary matrices. Define weight of passage as  $W : \Omega_n \rightarrow M_2(\mathbb{C})$  such that for  $\xi = (\xi_n, \dots, \xi_1) \in \Omega_n$ ,

$$W(\xi) = P_{\xi_n} \cdots P_{\xi_1} \quad (1.1)$$

with  $P_j = \Pi_j U$ , where  $\Pi_j$  is projection onto  $\mathbf{e}_j$ . Here  $M(2)$  is the set of all the complex-valued  $2 \times 2$  matrices. In this paper, we consider bi-product  $n$ -truncated path space  $\Omega_n^2 = \Omega_n \times \Omega_n$ . The algebra of subsets of  $\Omega_n^2$  is denoted by  $\mathcal{F}_n = 2^{\Omega_n^2}$ . For fixed  $\phi \in \mathcal{H}_C$  with  $\|\phi\| = 1$  called initial coin state, we define  $\varphi_{\phi,n} : \mathcal{F}_n \rightarrow \mathbb{C}$  by for any  $A \in \mathcal{F}_n$ ,

$$\varphi_{\phi,n}(A) = \left\langle \phi, \sum_{(\xi,\eta) \in A} W(\xi)^\dagger \cdot W(\eta) \phi \right\rangle. \quad (1.2)$$

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If  $A = \emptyset$ , then  $\varphi_{\phi,n}(A) \equiv 0$  for the convenience. We should remark that the map  $\varphi_n$  expresses  $\mathbb{C}$ -valued measure on  $\mathcal{F}_n$  in the following sense: for every  $\phi \in \mathcal{H}_C$  with  $\|\phi\| = 1$ ,

**Property of  $\varphi_{\phi,n}$**

- (i) For  $A \in \mathcal{F}_n$ ,  $\varphi_{\phi,n}(A) \in \mathbb{C}$ . Furthermore,  $\varphi_n(\Omega_n^2) = 1$ ,
- (ii) For any  $A_1, \dots, A_m \in \mathcal{F}_n$  with  $A_i \cap A_j = \emptyset$  ( $i \neq j$ ),

$$\varphi_{\phi,n} \left( \bigcup_{i=1}^m A_i \right) = \sum_{i=1}^m \varphi_{\phi,n}(A_i).$$

In particular, for  $\xi, \eta \in \Omega_n$ ,

$$(D)_{\xi,\eta} \equiv \varphi_{\phi,n}(\{(\xi, \eta)\})$$

is called the decoherence matrix starting from the initial coin state  $\phi$  which has been studied by [1, 2, 3]. Moreover for any  $A_0 \in 2^{\Omega_n}$ ,  $\nu_n(A_0) \equiv \varphi_n(A_0 \times A_0)$  is called  $q$ -measure on  $2^{\Omega_n}$  [1, 2].

Let  $\Omega^2 = \Omega_n^2 \times \{-1, 1\}^2 \times \{-1, 1\}^2 \times \dots = (\{-1, 1\}^2)^{\mathbb{N}}$ . A subset  $A \subset \Omega^2$  is a cylinder set if and only if there exist  $n \in \{1, 2, \dots\}$  and  $B \in \mathcal{F}_n$  such that  $A = B \times \{-1, 1\}^2 \times \{-1, 1\}^2 \times \dots$ . Denote  $\mathcal{C}(\Omega^2)$  as the collection of all cylinder sets. From the unitarity of  $U = P_1 + P_{-1}$ , we see that for  $A \in \mathcal{F}_n$ ,

$$\varphi_{\phi,n+1}(A \times \{-1, 1\}^2) = \left\langle \phi, \sum_{(\xi, \eta) \in A} \{(P_1 + P_{-1})W(\xi)\}^\dagger \cdot \{(P_1 + P_{-1})W(\eta)\} \phi \right\rangle = \varphi_{\phi,n}(A).$$

Thus if  $A \in \mathcal{F}_n$ , then

$$\varphi_{\phi,n+m}(A \times \{-1, 1\}^{2m}) = \varphi_{\phi,n}(A), \quad (1.3)$$

for any  $m \geq 1$ . Define  $\varphi_\phi : \mathcal{C}(\Omega^2) \rightarrow \mathbb{C}$  such that for any  $A \in \mathcal{C}(\Omega^2)$  expressed by  $A = B \times \{-1, 1\}^2 \times \{-1, 1\}^2 \times \dots$  with  $B \in \mathcal{F}_n$ ,

$$\varphi_\phi(A) = \varphi_{\phi,n}(B).$$

Equation (1.3) implies that if  $B = B_1 \times \{-1, 1\}^{2(n-m)}$  with  $B_1 \in \mathcal{F}_m$  and  $m \leq n$ , then  $\varphi_\phi(A) = \varphi_{\phi,n}(B) = \varphi_{\phi,m}(B_1)$ . So  $\varphi_\phi$  is well defined. Moreover, we easily find that  $\varphi_\phi$  satisfies both properties (i) and (ii).

For an initial coin state  $\phi$ , the *usual* quantum walk at time  $n$ ,  $X_n^{(\phi)}$ , originated by Gudder (1988) [4], is reexpressed by using  $\varphi_\phi$  as follows. For  $j \in \mathbb{Z}$ , and  $n \in \mathbb{N}$ , define  $T_n^{(j)} \in \mathcal{C}(\Omega^2)$  as  $T_n^{(j)} = \{(\xi, \eta) \in \Omega^2 : \xi_1 + \dots + \xi_n = \eta_1 + \dots + \eta_n = j\}$ , where  $\Omega = \{-1, 1\}^{\mathbb{N}}$ . Indeed, we can check that  $\varphi_\phi(T_n^{(j)}) \geq 0$ , and  $\sum_{j \in \mathbb{Z}} \varphi_\phi(T_n^{(j)}) = 1$ . The property (i) implies that

$$\varphi_\phi \left( \Omega_n^2 \setminus \bigcup_{j=-n}^n T_n^{(j)} \right) = 0.$$

Anyway, under the subalgebra  $2^{\bigcup_{j=-n}^n T_n^{(j)}} \subset 2^{\Omega_n^2}$ , the quantum walk at time  $n$  is denoted by a random variable  $X_n^{(\phi)} : \bigcup_{j=-n}^n T_n^{(j)} \rightarrow \{-n, -(n-1), \dots, n-1, n\}$ . Here  $X_n^{(\phi)}(\xi, \eta) = \xi_1 + \dots + \xi_n = \eta_1 + \dots + \eta_n$  has the following distribution:

$$P(X_n^{(\phi)} = j) \equiv P\left(\left\{(\xi, \eta) \in \bigcup_{j=-n}^n T_n^{(j)} : X_n^{(\phi)}(\xi, \eta) = j\right\}\right) = \varphi_\phi(T_n^{(j)}).$$

This is an equivalent expression for the definition of the usual quantum walk on  $\mathbb{Z}$  which has been intensively studied by many researchers. Now using  $\varphi_\phi$ , we can measure various kinds of cylinder sets including  $T_n^{(j)}$  corresponding to the usual quantum walk. In the next section, we choose three kinds of  $n$ -truncated cylinder sets in  $\mathcal{C}(\Omega^2)$  by using our measure  $\varphi_\phi$  and find their asymptotics for large  $n$ .

## 2 Results

For  $x, y \in \mathbb{R}$ , define the set of all paths which go through the positions  $nx$  and  $ny$  at time  $n$  and  $2n$ , respectively as follows:

$$\Theta_{x|y}^{(n)} = \left\{(\xi_1, \xi_2, \dots) \in \Omega : \frac{\xi_1 + \dots + \xi_n}{n} = x, \frac{\xi_1 + \dots + \xi_{2n}}{n} = y\right\}.$$

Now we concentrate on  $y = 0$ , and the following three cases with respect to the pair of  $\Theta_x^{(n)} \times \Theta_y^{(n)} \in \mathcal{C}(\Omega^2)$ , where  $\Theta_x^{(n)} \equiv \Theta_{x|0}^{(n)}$ :

- (1)  $A_1^{(n)} \equiv \bigcup_{x \in \mathbb{R}} \bigcup_{y \in \mathbb{R}} \Theta_x^{(n)} \times \Theta_y^{(n)}$  case
- (2)  $A_2^{(n)}(y) \equiv \bigcup_{x \in \mathbb{R}} \Theta_x^{(n)} \times \Theta_y^{(n)}$  with fixed  $y \in \mathbb{R}$  case
- (3)  $A_3^{(n)}(y) \equiv \Theta_y^{(n)} \times \Theta_y^{(n)}$  case

Note that  $A_1^{(n)} \supseteq A_2^{(n)} \supseteq A_3^{(n)}$ . To explain the situations of  $A_j^{(n)}$ 's, we prepare two quantum walkers, walker 1 and walker 2, who produce the weight of path  $W$ . The measurement value is obtained by inner product of their weight of paths with an initial coin state (see Eq. (1.2)). Both walkers in  $A_1^{(n)}$  give weight of all the paths returning back to the origin at time  $2n$ . Walkers 1 and 2 in  $A_3^{(n)}$  produce weight of every return path with length  $2n$  restricted to passing the position  $nx$  at time  $n$ . In  $A_2^{(n)}$ , despite of  $A_1^{(n)}$  and  $A_3^{(n)}$ , the classes of return paths for two walkers are different: walker 1 is in the situation (1) while walker 2 is in the situation (3).

The following theorem gives asymptotics of measurement value for each situation (1)-(3) by using  $\varphi_{\phi, n}$ . Define

$$\mathcal{D}_\kappa = e^{i\kappa} \Pi_{-1} + \Pi_1 \quad \text{with } \kappa = \arg(a) + \arg(c) - \det(U). \quad (2.4)$$

We use notation  $a_n \sim b_n$  as  $\lim_{n \rightarrow \infty} a_n/b_n = 1$ .

**Lemma 1** Denote the Konno function  $f_K(x; r)$  ( $0 < r < 1$ ) [9, 10] by

$$f_K(x; r) = \frac{\sqrt{1-r^2}}{\pi(1-x^2)\sqrt{r^2-x^2}} \mathbf{1}_{\{|x|<r\}}(x),$$

where  $\mathbf{1}_A(x)$  is the indicator function, that is,  $\mathbf{1}_A(x) = 1$ , ( $x \in A$ ),  $= 0$ , ( $x \notin A$ ). Let the initial coin state be  $\phi_0$ , and  $\phi_\kappa \equiv \mathcal{D}_\kappa \phi_0$ . Then we have for large  $n$ ,

(1) Case (1)

$$\varphi_{\phi_0}(A_1^{(n)}) \sim \frac{f_K(0; |a|)}{n} = \frac{|c|}{\pi|a|n}. \quad (2.5)$$

(2) Case (2)

$$\sum_{j:j < ny} \varphi_{\phi_0}(A_2^{(n)}(j/n)) \sim \mathbf{1}_{\{y>0\}}(y) \frac{f_K(0; |a|)}{n} = \mathbf{1}_{\{y>0\}}(y) \frac{|c|}{\pi|a|n}. \quad (2.6)$$

(3) Case (3)

$$\sum_{j \leq ny} \varphi_{\phi_0}(A_3^{(n)}(j/n)) \sim \frac{|c|^2}{|a|^2 n} \int_{-\infty}^y (1 + \langle \phi_\kappa, C_0 \phi_\kappa \rangle x) \frac{\mathbf{1}_{\{|x|<|a|\}}(x)}{\pi^2(1-x^2)^2} dx, \quad (2.7)$$

where

$$C_0 = \begin{bmatrix} 1 & -|c|/|a| \\ -|c|/|a| & -1 \end{bmatrix}.$$

Now we present a distribution function with respect to  $q$ -measure [1, 2]. To do so, put

$$F_{n,\phi_0}(x|y) \equiv \frac{\sum_{j \leq nx} \varphi_{\phi_0}(\Theta_{(j/n)|y}^{(n)} \times \Theta_{(j/n)|y}^{(n)})}{\sum_{j \leq n} \varphi_{\phi_0}(\Theta_{(j/n)|y}^{(n)} \times \Theta_{(j/n)|y}^{(n)})}.$$

We can easily check that for fixed  $y$ ,  $F_{n,\phi_0}(x|y)$  becomes a distribution function, that is,

- (a)  $\lim_{x \rightarrow \infty} F_{n,\phi_0}(x|y) = 1$ ,  $\lim_{x \rightarrow -\infty} F_{n,\phi_0}(x|y) = 0$ ,
- (b) for any  $x \leq y$ ,  $0 \leq F_{n,\phi_0}(x|z) \leq F_{n,\phi_0}(y|z) \leq 1$ .

The function  $F_{n,\phi_0}(x|y)$  corresponds to a *normalized*  $q$ -measure [1, 2] restricted to the event  $\bigcup_x \Theta_{x|y}^{(n)}$ . Part (3) in Lemma 1 leads the following theorem for  $y = 0$  case:

**Theorem 1** Assume that the initial coin state is  $\phi_0 = {}^T[\alpha, \beta]$ . We consider the sequence  $\{F_{n,\phi_0}(x|0)\}_{n \geq 0}$ . Let  $Y_n$  be a random variable whose distribution function is  $F_{n,\phi_0}(x|0)$ , that is,  $P(Y_n \leq x) = F_{n,\phi_0}(x|0)$ . Then we have

$$Y_n \Rightarrow Z, \quad (n \rightarrow \infty) \quad (2.8)$$

where  $Z$  has the following density:

$$\nu_{\phi_0}(x|0) = \frac{|c|^2}{|a| + |c|^2 \log \sqrt{\frac{1+|a|}{1-|a|}}} \left[ 1 - \left\{ (|\alpha|^2 - |\beta|^2) + \frac{a\alpha\bar{b}\bar{\beta} + \bar{a}\bar{\alpha}b\beta}{|a|^2} \right\} x \right] \frac{\mathbf{1}_{\{|x|<|a|\}}(x)}{\pi^2(1-x^2)^2}$$

Here “ $\Rightarrow$ ” means the weak convergence.

Next, define  $W_x^{(n)} = \bigcup_y \Theta_{y|x}^{(n)}$  and

$$\widehat{G}_{n,\phi_0}(y|x) = \frac{\sum_{j \leq ny} \varphi_{\phi_0} \left( W_x^{(n)} \times \Theta_{(j/n)|x}^{(n)} \right)}{\sum_{j \leq n} \varphi_{\phi_0} \left( W_x^{(n)} \times \Theta_{(j/n)|x}^{(n)} \right)}.$$

The value  $\widehat{G}_{n,\phi_0}(y|x)$  satisfies the above condition (a), but the condition (b) is not ensured, that is,

$$\lim_{y \rightarrow \infty} \widehat{G}_{n,\phi_0}(y|x) = 1, \text{ and } \lim_{y \rightarrow -\infty} \widehat{G}_{n,\phi_0}(y|x) = 0,$$

while  $\widehat{G}_{n,\phi_0}(y|x) \in \mathbb{C}$  for  $|y| < \infty$  in general. From Parts (1) and (2) in Theorem 1, we obtain an asymptotic behavior of the value  $\widehat{G}_{n,\phi_0}(x|0)$  which is deeply related to the weak value [6, 7] as follows.

Before we show the result, here we briefly give the definition of the weak value. We can see more detailed explanations and its interesting related works in [8] and its references. Let  $\mathcal{H}$  be a Hilbert space and  $U(t_2, t_1)$  be an evolution from time  $t_1$  to  $t_2$  on  $\mathcal{H}$ . For an observable  $A$  and normalized states  $\phi_i, \phi_f \in \mathcal{H}$ , the weak value  ${}_{\phi_f} \langle A \rangle_{\phi_i}^w$  is defined by

$${}_{\phi_f} \langle A \rangle_{\phi_i}^w = \frac{\langle \phi_f, U(t_f, t) A U(t, t_i) \phi_i \rangle}{\langle \phi_f, U(t_f, t_i) \phi_i \rangle}. \quad (2.9)$$

Here  $\phi_i$  and  $\phi_f$  are called pre-selected state and post-selected state, respectively.

From now on, we take the Hilbert space  $\mathcal{H}$  as  $\bigoplus_{x \in \mathbb{Z}} \mathcal{H}_x$ , where  $\mathcal{H}_x$  is the two-dimensional Hilbert space spanned by left and right chiralities  $\{\mathbf{e}_L, \mathbf{e}_R\}$ . Let the canonical basis of  $\mathcal{H}$  be denoted by  $\{\delta_x \otimes \mathbf{e}_L, \delta_x \otimes \mathbf{e}_R; x \in \mathbb{Z}\}$ . Put a permutation operator  $S$  on  $\mathcal{H}$  such that for  $\delta_x \otimes \mathbf{e}_J$  ( $J \in \{L, R\}$ ),

$$S(\delta_x \otimes \mathbf{e}_J) = \begin{cases} \delta_{x+1} \otimes \mathbf{e}_R, & (J = R), \\ \delta_{x-1} \otimes \mathbf{e}_L, & (J = L). \end{cases}$$

Define  $E = SC$  be a unitary operator on  $\mathcal{H}$ , where  $C = \sum_x \oplus U$ . (Recall that  $U$  is the two-dimensional unitary operator.) We consider the iteration of  $E$  from the initial state  $\Phi_0 = \delta_0 \otimes \phi$  with  $\|\phi\|^2 = 1$ :

$$\Phi_0 \xrightarrow{E} \Phi_1 \xrightarrow{E} \Phi_2 \xrightarrow{E} \dots$$

This is another equivalent expression for the quantum walk on  $\mathbb{Z}$  with initial state  $\Phi_0$ . Indeed,

$$\|\Pi_j E^n \Phi_0\|^2 = \varphi_\phi(T_n^{(j)}),$$

where  $\Pi_j$  is the projection onto  $\mathcal{H}_j$ .

In particular, when we take for  $t_1, t_2 \in \mathbb{N}$ ,  $E^{t_2-t_1}$  as  $U(t_2, t_1)$  and  $\Pi_j$  as the observable  $A$ , moreover  $\Phi_0$  as the pre-selected state and  $\Pi_0 \overline{U}^{t_f} \Phi_0$  as the post-selected state in Eq. (2.9) with  $t_i = 0$ ,  $t = n$  and  $t_f = 2n$ , then we have in this setting

$$\sum_{j \leq ny} {}_{\phi_f} \langle A \rangle_{\phi_i}^w = \widehat{G}_{n,\phi_0}(y|0). \quad (2.10)$$

This is a connection between our complex-valued measure and weak value. We find that the weak value weakly converges to the delta measure as follows.

**Theorem 2** *It is hold that for large  $n$ ,*

$$\lim_{n \rightarrow \infty} \widehat{G}_{n, \phi_0}(y|0) = \mathbf{1}_{\{y>0\}}(y). \quad (2.11)$$

The physical meaning of Theorem 2 remains as an interesting open problem.

### 3 Proof of Lemma 1

Let  $\Xi_n(j) = \sum_{\xi: \xi_n + \dots + \xi_1 = j} W(\xi)$  be weight of all the  $n$ -truncated passages arriving at  $j$ . Our proof is based on the stationary phase method:

**Lemma 2** *Let  $f(x)$  denote an  $\mathbb{R}$ -valued function on  $[a, b]$  satisfying that there exists a unique  $c \in [a, b]$  such that  $f'(c) = 0$  with  $f''(c) \neq 0$ . Then for large  $n$ ,*

$$\frac{1}{n} \sum_{j: an < j < bn} g(j/n) e^{inf(j/n)} \sim e^{i \operatorname{sgn}(f''(c))\pi/4} \sqrt{\frac{2\pi}{|f''(c)|n}} g(c) e^{inf(c)} + o(1/\sqrt{n}), \quad (3.12)$$

where for  $y \in \mathbb{R}$ ,  $\operatorname{sgn}(y) = 1, (y > 0), = 0, (y = 0), = -1, (y < 0)$ .

At first we give the following key lemma whose proof is described in Appendix by using the stationary phase method:

**Lemma 3** *Put  $\mathbb{R}$ -valued functions  $k(x)$  and  $\psi(x)$  ( $x \in [-|a|, |a|]$ ) as*

$$e^{ik(x)} = \frac{1}{|a|} \sqrt{\frac{|a|^2 - x^2}{1 - x^2}} + i \frac{|c|}{|a|} \frac{x}{\sqrt{1 - x^2}}, \quad (3.13)$$

$$e^{i\psi(x)} = \sqrt{\frac{|a|^2 - x^2}{1 - x^2}} + i \frac{|c|}{\sqrt{1 - x^2}}. \quad (3.14)$$

For any  $j \in \mathbb{Z}$  with  $j = nx$  ( $x \in [-1, 1]$ ), we obtain

$$\begin{aligned} \Xi_n(j) &= \frac{1 + (-1)^{n+j}}{2} e^{in\delta/2} \sqrt{\frac{2f_K(x; |a|)}{n}} \\ &\times \mathcal{D}_\kappa^\dagger \left( e^{i\pi/4} e^{in(\psi(x) - xk(x))} \Pi(x) + e^{-i\pi/4} e^{-in(\psi(x) - xk(x))} \overline{\Pi(x)} \right) \mathcal{D}_\kappa + o(1/\sqrt{n}), \end{aligned} \quad (3.15)$$

where

$$\Pi(x) = \begin{bmatrix} |a|(1-x) & |c|x + i\sqrt{|a|^2 - x^2} \\ |c|x - i\sqrt{|a|^2 - x^2} & |a|(1+x) \end{bmatrix}.$$

Here for  $M \in M_2(\mathbb{C})$ ,  $\left(\overline{M}\right)_{i,j} = \overline{(M)_{i,j}}$  for any  $i, j \in \{1, 2\}$ .

Before the proof of Lemma 1, we can confirm a consistency of the statement of the above lemma as follows. Recall that  $X_n^{(\phi)}$  is a random variable determined by  $P(X_n^{(\phi)} = j) = \|\Xi_n(j)\phi\|^2$  with the initial coin state  $\phi = [\alpha, \beta]$  so called usual quantum walk. Then Lemma 3 and the Riemann-Lebesgue lemma imply the following corollary with respect to  $X_n^{(\phi)}$ :

### Corollary 3

$$\lim_{n \rightarrow \infty} P(X_n^{(\phi)}/n \leq x) = \int_{-\infty}^x \left\{ 1 - \left( |\alpha|^2 - |\beta|^2 + \frac{2\operatorname{Re}[a\alpha\bar{b}\beta]}{|a|^2} \right) x \right\} f_K(x; |a|) dx.$$

This is consistent with results of [9, 10]. Now we give the proof of Lemma 1 in the following:

(1) *Proof of Case (1).*

Put  $g(x) = \psi(x) - xk(x)$ . We should remark that  $L_n(x) \equiv \sum_{\xi \in \Theta_x^{(n)}} W(\xi) = \Xi_n(-nx)\Xi_n(nx)$ . Note that  $\sum_{j=-n}^n L_n(j/n) = \Xi_{2n}(0)$ . Lemma 3 reduces to

$$e^{-in\delta/2} D_\kappa \Xi_{2n}(0) D_\kappa^\dagger \sim \sqrt{\frac{f_K(0; |a|)}{n}} \left\{ e^{i\frac{\pi}{4}} e^{2ing(0)} \Pi(0) + e^{-i\frac{\pi}{4}} e^{-2ing(0)} \overline{\Pi(0)} \right\}. \quad (3.16)$$

By using the fact that for every  $x \in \mathbb{R}$ ,

$$\Pi^2(x) = \Pi(x), \quad \Pi(x) \overline{\Pi(-x)} = 0, \quad (3.17)$$

and Eq. (3.16), we obtain

$$\varphi_{\phi_0}(A_1^{(n)}) = \sum_{i=-n}^n \sum_{j=-n}^n \langle L_n(i/n) \phi_0, L_n(j/n) \phi_0 \rangle \quad (3.18)$$

$$= \langle \Xi_{2n}(0) \phi_0, \Xi_{2n}(0) \phi_0 \rangle \quad (3.19)$$

$$\sim \frac{f_K(0; |a|)}{n} \left\langle \phi_0, \left\{ \Pi(0) + \overline{\Pi(0)} \right\} \phi_0 \right\rangle = \frac{f_K(0; |a|)}{n}. \quad (3.20)$$

Then we complete the proof of case (1). It is consistent with the result of [11] which treats the Hadamard walk.

□

(2) *Proof of Case (2).*

Using Eq. (3.17), Lemma 3 implies that

$$D_\kappa \Xi_n(-j) \Xi_n(j) D_\kappa^\dagger e^{-in\delta} \sim i \frac{1 + (-1)^{n+j}}{2} \times \frac{2f_K(x; |a|)}{n} \times \left\{ e^{2ing(x)} \Pi(-x) \Pi(x) - e^{-2ing(x)} \overline{\Pi(-x) \Pi(x)} \right\}, \quad (3.21)$$

By Eq. (3.16),

$$\begin{aligned} & e^{-in\delta} \sum_{j < ny} D_\kappa \Xi_n(-j) \Xi_n(j) D_\kappa^\dagger \\ & \sim \frac{2i}{n} \sum_{j < ny} f_K(j/n; |a|) \left\{ e^{2ing(j/n)} \Pi(-j/n) \Pi(j/n) - e^{-2ing(j/n)} \overline{\Pi(-j/n) \Pi(j/n)} \right\} \end{aligned} \quad (3.22)$$

Now we consider the solution for  $g'(x) = \psi'(x) - k(x) - xk'(x) = 0$ . Equations (A.30)-(A.33) in Appendix imply that  $\psi(x) = \theta(k(x))$  and  $k(x)$  is the unique solution for

$$h(k) = \partial\theta(k)/\partial k = x \quad (3.23)$$

on  $k \in [-\pi/2, \pi/2]$ , where  $\cos \theta(k) = |a| \cos k$  with  $\sin \theta(k) \geq 0$ . So we have

$$\frac{\partial\psi(x)}{\partial x} = \frac{\partial\theta(k(x))}{\partial x} = xk'(x).$$

Then we obtain

$$g'(x) = -k(x). \quad (3.24)$$

On the other hand, differentiating both sides of Eq. (3.23) with respect to  $x$  implies

$$\frac{\partial}{\partial x} \left( \frac{\partial\theta(k)}{\partial k} \right) = \frac{\partial k}{\partial x} \left( \frac{\partial^2\theta(k)}{\partial k^2} \right) = 1.$$

Then Eq. (3.24) gives

$$k'(x) = \frac{1}{\partial^2\theta(k)/\partial k^2} \Big|_{k=k(x)} = \pi f_K(x; |a|). \quad (3.25)$$

Thus,  $g'(x) = 0$  if and only if  $k(x) = 0$ , which implies  $e^{ik(x)} = 1$ . Therefore by definition of  $k(x)$  (see Eq. (3.13)),  $x = 0$  is the unique solution for  $g'(x) = 0$ . Moreover Eqs. (3.24) and (3.25) give  $g''(x) = -k'(x) = -\pi f_K(x; |a|)$ , which implies  $g''(0) = -\pi f_K(0; |a|)$ . So applying the stationary phase method described in Lemma 2 to Eq. (3.22), we obtain

$$\begin{aligned} & e^{-in\delta} \sum_{j < ny} D_\kappa \Xi_n(-j) \Xi_n(j) D_\kappa^\dagger \\ & \sim i \mathbf{1}_{\{y > 0\}}(y) \left( e^{2in\psi(0)} e^{-i\pi/4} \Pi(0) - e^{-2in\psi(0)} e^{i\pi/4} \overline{\Pi(0)} \right) \sqrt{\frac{f_K(0; |a|)}{n}}. \end{aligned} \quad (3.26)$$

Combining Eq. (3.26) with Eq. (3.16), we arrive at

$$\varphi_{\phi_0}(A_2^{(n)}(y)) = \sum_{j: j < ny} \langle \Xi_{2n}(0) \phi_0, \Xi_n(-j) \Xi_n(j) \phi_0 \rangle \sim \mathbf{1}_{\{y > 0\}}(y) \frac{f_K(0; |a|)}{n}. \quad (3.27)$$

So we complete the proof. □

### (3) Proof of Case (3).

Remark that

$$\sum_{j \leq ny} \varphi_{\phi_0}(A_3^{(n)}(j/n)) = \sum_{j \leq ny} \langle L_n(j/n) \phi_0, L_n(j/n) \phi_0 \rangle. \quad (3.28)$$

On the other hand, using the relations of  $\Pi(x)$  described by Eq. (3.17), Eq. (3.21) gives

$$L_n^\dagger(x) \cdot L_n(x) \sim \frac{|c|^2}{n^2 |a|^2} (I + C_0 x) \frac{\mathbf{1}_{\{|x| < |a|\}}(x)}{\pi^2 (1 - x^2)^2} \quad (3.29)$$

which leads the desired conclusion of case (3).



□

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## A Proof of Lemma 3

We take the spatial Fourier transform of the weight of path  $\Xi_n(j)$  such that

$$\widehat{\Xi}_n(k) = \sum_{j \in \mathbb{Z}} \Xi_n(j) e^{ijk}.$$

From the recurrence relation  $\Xi_{n+1}(j) = Q\Xi_n(j-1) + P\Xi_n(j+1)$ , we obtain

$$\widehat{\Xi}_n(k) = (e^{ik}Q + e^{-ik}P)^n.$$

The eigenvalues and their corresponding normalized eigenvectors are expressed by  $\lambda_m(k+\tau)$  and  $\mathbf{v}_m(k+\tau)$ , ( $m \in \{0, 1\}$ ), where

$$\lambda_m(k) = e^{i\delta/2} \cdot e^{i(-1)^m\theta(k)}, \quad (\text{A.30})$$

$$\mathbf{v}_m(k) = \frac{1}{\sqrt{2\{1 - |a| \cos[(-1)^m\theta(k) - k]\}}} D_\kappa^\dagger \left[ |a| - e^{i((-1)^m\theta(k) - k)} \right], \quad (\text{A.31})$$

where  $\tau = \delta/2 - \arg(a)$  and  $D_\kappa$  is defined in Eq. (2.4). Here  $\cos\theta(k) = |a| \cos k$  with  $\sin\theta(k) \geq 0$  and  $\delta = \arg(\det(U))$ . By the Fourier inversion theorem, we obtain for any  $\gamma \in \mathbb{R}$ ,

$$\begin{aligned} \Xi_n(j) &= \int_\gamma^{2\pi+\gamma} \widehat{\Xi}_n(k) e^{-ij k} \frac{dk}{2\pi}, \\ &= e^{in\delta/2} \sum_{m \in \{0,1\}} \int_{\gamma+\tau}^{2\pi+\gamma+\tau} e^{in((-1)^m\theta(k) - xk)} \mathbf{v}_m(k) \mathbf{v}_m(k)^\dagger \frac{dk}{2\pi}, \end{aligned} \quad (\text{A.32})$$

where  $x = j/n$ . We choose an arbitrary parameter  $\gamma$  as  $-\tau - \pi/2$ . From now on we apply the stationary phase method in Lemma 2 to Eq. (A.32). Put  $f_m(k) = (-1)^m\theta(k) - xk$ , ( $m \in \{0, 1\}$ ) as  $\mathbb{R}$ -valued function on  $[-\pi/2, 3\pi/2)$ . The solution for  $\partial f_m(k)/\partial k = 0$  is given by

$$(-1)^m h(k) = x, \quad (\text{A.33})$$

where  $h(k) = \partial\theta(k)/\partial k$ . In the following, we consider  $m = 0$  case. The definition of  $\theta(k)$  gives

$$h(k) = \frac{|a| \sin k}{\sqrt{1 - |a|^2 \cos^2 k}}.$$

The solutions for  $h'(k) = 0$  in  $[-\pi/2, 3\pi/2)$  are  $\pm\pi/2$ . We denote  $h_\pm(k)$  with  $h(k) = h_+(k) + h_-(k)$  so that  $h'_+(k) > 0$  and  $h_-(k) \leq 0$ , as the function on  $K_+ = [-\pi/2, \pi/2)$  and  $K_- = [\pi/2, 3\pi/2)$ , respectively. To apply the stationary phase method, we divide the integral in Eq. (A.32) into the four parts as follows:

$$e^{-in\delta/2} D_\kappa \Xi_n(j) D_\kappa^\dagger = \sum_{m \in \{0,1\}} \sum_{\epsilon \in \{-, +\}} \int_{k \in K_\epsilon} e^{in((-1)^m\theta(k) - xk)} \mathbf{v}_m(k) \mathbf{v}_m(k)^\dagger \frac{dk}{2\pi}. \quad (\text{A.34})$$

An explicit expression for the solutions  $k_\pm(x)$  for  $h_\pm(k) = x$ , respectively, are obtained as follows:

$$\cos k_\pm(x) = \pm \frac{1}{|a|} \sqrt{\frac{|a|^2 - x^2}{1 - x^2}}, \quad (\text{A.35})$$

$$\sin k_\pm(x) = \frac{|c|}{|a|} \frac{x}{\sqrt{1 - x^2}}. \quad (\text{A.36})$$

Thus we have

$$\left| \frac{1}{\partial^2 f_0(k)/\partial k^2} \right|_{k=k_{\pm}(x)} = \left| \frac{1}{\partial h(k)/\partial k} \right|_{k=k_{\pm}(x)} = \pi f_K(x; |a|). \quad (\text{A.37})$$

Moreover some algebraic computations give

$$\begin{aligned} \mathbf{v}_0(k)\mathbf{v}_0(k)^\dagger|_{k=k_+(x)} &= D_\kappa^\dagger \Pi(x) D_\kappa, & \mathbf{v}_0(k)\mathbf{v}_0(k)^\dagger|_{k=k_-(x)} &= D_\kappa^\dagger \overline{\Pi(x)} D_\kappa, \\ \mathbf{v}_1(k)\mathbf{v}_1(k)^\dagger|_{k=k_+(x)} &= D_\kappa^\dagger \overline{\Pi(-x)} D_\kappa, & \mathbf{v}_1(k)\mathbf{v}_1(k)^\dagger|_{k=k_-(x)} &= D_\kappa^\dagger \Pi(-x) D_\kappa. \end{aligned} \quad (\text{A.38})$$

For the solutions of Eq. (A.33) in  $m = 1$  case, we replace the parameter  $x$  in the result on  $m = 0$  case given by the above discussion with  $-x$ . By putting  $\psi(x)$  as  $\psi(x) = \theta(k(x))$  with  $k(x) \equiv k_+(x)$ , note that  $\theta(k_+(-x)) = \psi(x)$ ,  $\theta(k_-(x)) = \pi - \psi(x)$ , and  $k_+(-x) = -k(x)$ ,  $k_-(x) = -k(x) - \pi$ . Inserting these relations and Eqs. (A.37) and (A.38) into the formula in Lemma 2 for each term  $(\epsilon, m) \in \{(+, 0), (+, 1), (-, 0), (-, 1)\}$  in Eq. (A.34), we have the desired conclusion.

□